Reply to "Comment on 'Liénard systems, limit cycles, and Melnikov theory' "

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The problem of finding the number of limit cycles of Liénard systems, $\dot{x}=y-\varepsilon F(x,\mu)$, $\dot{y}=-x$, where $F(x,\mu)$ is an odd polynomial, was addressed by Giacomini and Neukirch [Phys. Rev. E **56**, 3809 (1998)], and they proposed an original method, where a sequence of polynomials is introduced, whose roots give the number of limit cycles that also allow one to construct a sequence of algebraic approximations to the limit cycles. This author showed [Phys. Rev. E **57**, 340 (1998)] that in the limit of these sequences, the same information is given by a polynomial which Melnikov theory associates to each Liénard system. In their comment, Giacomini and Neukirch [Phys. Rev. E **59**, 2483 (1999)] remark that this is correct only for small values of ε. I wish to stress here that this is right and that the reason for it lies in the perturbative nature of Melnikov theory, while the Giacomini and Neukirch method is nonperturbative. As a consequence the original conjecture is reformulated. [S1063-651X(98)12612-0]

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The general problem of the number and location of limit cycles in Liénard systems is the main objective of the algorithm developed by Giacomini and Neukirch in their paper [1]. The publication of this interesting method has stimulated research [2–5] concerning the problem of the number of limit cycles in Liénard systems, a problem that, as is noticed in Refs. [2,7], is somehow related to the already unsolved 16th Hilbert problem.

The main idea of the method consists in determining a sequence of polynomials $R_n(x,\varepsilon)$ for a given Liénard system of the type

$$\dot{x}=y-\varepsilon F(x,\mu),\qquad \dot{y}=-x, \tag{1}$$

where $F(x,\mu)$ is an odd polynomial in x, and ε and μ are parameters. The number of limit cycles is given by the roots of these polynomials. Furthermore, they constructed some Lyapunov-like functions $h_n(x,y,\varepsilon)$ which are related to another sequence of polynomials $g_{j,n}(x,\varepsilon)$, $0\leqslant j\leqslant n-1$, through which a sequence of algebraic approximations to the limit cycles is constructed. In addition, they made a conjecture on the relationship between the number of limit cycles and the roots of the $R_n(x,\varepsilon)$ polynomials, a problem which has attracted the attention of Llibre *et al.* in Ref. [5], where a discussion of this conjecture appears.

In Ref. [2] the author of this Reply argues that the same problem addressed by Giacomini and Neukirch is connected to the theory of Melnikov. In fact it is pointed out that with the help of this theory, a polynomial, called the Melnikov polynomial, is associated with a Liénard system, giving the same results as the Giacomini-Neukirch (GN) polynomials [8] for the examples shown in Ref. [1].

Reference [2] tries also to give a hint for the proof of the conjecture given by the authors of [1] using Melnikov theory and another conjecture is stated in relation to the behavior of the roots of the Melnikov polynomial and its relationship to the roots of the GN polynomials. Moreover, it was noticed that the result of the integration of $R_n(x,\varepsilon)$ along the limit cycle of period $T=2\pi$, for the case of the van der Pol oscillator, gives another polynomial that for the first iterations coincided with the radius of the limit cycle given by the Melnikov polynomial. This idea has been successfully developed in Ref. [3], improving notably the previous results given in Ref. [1].

The main idea of the comment by Giacomini and Neukirch [6] is to make clear, providing explicit numerical evidence, that what is claimed in Ref. [2] is only correct for very small values of ε. In Refs. [1,2] $\varepsilon=1$, and the true dependence on ε of the polynomials associated to the Liénard systems is not made clear enough. In particular, the GN polynomials depend on ε and on μ, however, the Melnikov polynomial only depends on μ. This results from the fact that the Melnikov function is constructed assuming that what multiplies ε in Eq. (1) constitutes a small perturbation (in fact, a first order perturbation) to a Hamiltonian system [with $H(x,y)=(y^2+x^2)/2$ in our case], however, the nature of the Giacomini and Neukirch algorithm is nonperturbative.

Consequently, I wish to stress here that what is claimed by Giacomini and Neukirch in their Comment is absolutely right, which is consistent with the perturbative nature of the Melnikov theory.

Thus the main conclusion is that both theories are correct for small values of ε, or in other words, the GN polynomials and the Melnikov polynomial associated to the Liénard systems give the same results provided that $\varepsilon\to 0$. This might help to prove their conjecture for small values of ε. Moreover, the conjecture which appears at the end of Ref. [2], should be reformulated and written as follows.

Conjecture: For a given Liénard system, Eq. (1), there are associated a Melnikov polynomial $P(r^2,m)$ [9], and two sequences of polynomials $R_n(x,\varepsilon)$ and $g_{1,n}(x,\varepsilon)$. For a fixed given value of m, each positive root of $P(r^2,m)$, α,

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is related to a root of $R_n(x,\varepsilon)$, $\alpha_n(\varepsilon)$, and to a root of $g_{1,n}(x,\varepsilon)$, $\beta_n(\varepsilon)$, such that $\alpha_n(\varepsilon)<\alpha<\beta_n(\varepsilon)$, and with the property that for $\varepsilon\to 0$ and as n increases, $\alpha_n(\varepsilon)\to\alpha$ and $\beta_n(\varepsilon)\to\alpha$. (Note that it is understood here that the limit $\varepsilon\to 0$ needs to be taken first and then the limit $n\to\infty$, otherwise the conjecture is not valid.)

Nevertheless it is important to note that the GN polynomials have no limitation on the parameter ε, since it is a nonperturbative theory, and this is the main advantage of the Giacomini and Neukirch method.

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[7] T. R. Blows and L. M. Perko, SIAM (Soc. Ind. Appl. Math.) Rev. **36**, 341 (1994).

[8] The GN polynomials refer to the polynomials $R_n(x,\varepsilon)$ and $g_{j,n}(x,\varepsilon)$ related to the Lyapunov-like functional $h_n(x,y,\varepsilon)$, which Giacomini and Neukirch associate with each Liénard system, although the more relevant one is $R_n(x,\varepsilon) = (d/dt)[h_n(x,y,\varepsilon)]$.

[9] The Melnikov polynomial $P(r^2,m)$ is proportional to the Melnikov function associated with the Liénard system and is a polynomial of mth degree in r^2. For a particular case the positive root of this polynomial gives the radius of the limit cycle.